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Further studies of a simple gyrotron equation: linear theory

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Abstract

A linearized version of a standard system of gyrotron model equations is studied. The linearization allows the inclusion of some effects of particle bunching. The normal modes of the linearized system are given. It is shown that bunching effects couple incoming and outgoing waves. The waves near resonance duplicate well-known results. Without bunching and with a simple background profile function integral representations of solutions are given and discussed.

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1. Introduction

An earlier paper with other authors [1] considered a simple model for a gyrotron and applied more or less standard methods to analyze the steady state equations intended to represent the dynamics of the electrons and electromagnetic fields in the amplifier part of the gyrotron. It is the purpose of this paper to extend that analysis and study the dynamics of the linearized system. The model equations we use have been given many times before, and we cite only a text [2], a standard [3], and two earlier works which heavily influenced this study [4, 5]. We refer the reader to these sources for greater detail on the model and background on gyrotron dynamics.

Often the linearized version of a system of equations can give some qualitative indication of the nature of the solutions of the full system. To that end we first explore the class of background states about which to linearize. We assume that no electromagnetic field is present and we select a particular class of background states which allow some, but not all, effects of particle bunching. With a chosen background state it is straightforward to develop the linearized system in which one finds that the effects of particle bunching are reduced to one parameter. We study waves in a uniform state with and without bunching. In this analysis we identify a cut-off and a resonance and make contact with standard results, such as are found in [2]. For the case with no bunching and a particularly simple non-uniform background state we are able to give integral representations of solutions for the gyrotron model. We can analyze these solutions and extract physically relevant information. With the solutions we could presumably complete the solution for a more generalized background profile by means of geometrical optics and matched asymptotic expansions, although we do not carry out this intricate and far from trivial task.

The dimensionless model used in [1, 5] involved a complex-valued transverse electron momentum distribution function $p(z,\theta)$, where z is the axial coordinate and θ is the gyrophase angle and a complex-valued wave amplitude $f(z)$. A paraxial approximation is assumed and the real and imaginary parts of p and f are the x and y components of the electron momentum distribution function and the electric field. These functions satisfy the equation

$$\frac{\mathrm{d}p}{\mathrm{d}z} + \mathrm{i}(c + |p|^2)p = \mathrm{i}f(z) \tag{1}$$

and the cavity excitation equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \gamma(z)f = \frac{d}{2\pi}\oint_0^{2\pi} \mathrm{d}\theta p(z,\theta) \equiv d\langle p(z,\theta)\rangle. \tag{2}$$

In (2) $\gamma(z)$ is the square of the wave number of the free space waves in the gyrotron cavity at the position z. The Poynting theorem for the system is

$$\frac{d}{2\pi}\oint_0^{2\pi} \mathrm{d}\theta|p(z,\theta)|^2 + 2\mathrm{Im}\left(\frac{\mathrm{d}\bar{f}}{\mathrm{d}z}f\right) = \text{constant}. \tag{3}$$

In this system typically c and γ are of order 1 and d is small. It is convenient to rescale the system and introduce

$$\delta = d^{1/2} \tag{4}$$

$$f = \delta g \tag{5}$$

so that the system becomes

$$\frac{\mathrm{d}p}{\mathrm{d}z} + \mathrm{i}(c + |p|^2)p = \mathrm{i}\delta g \tag{6}$$

$$\frac{\mathrm{d}^2 g}{\mathrm{d}z^2} + \gamma(z)g = \delta\langle p\rangle \tag{7}$$

and the Poynting theorem is

$$\langle |p|^2\rangle + 2\mathrm{Im}(\bar{g}'g) = \text{const}. \tag{8}$$

With this scaling all quantities in the Poynting theorem are O(1) in δ. We note in passing that for a wave of the form $(\mathrm{e}^{-\mathrm{i}kz} + \alpha\,\mathrm{e}^{\mathrm{i}kz})$, where α and k are real, that the energy flux in the wave is just $k(1-\alpha^2)$. We use this observation to identify outgoing ($k > 0$) and incoming ($k < 0$) waves. We identify incoming and outgoing relative to a source or sink at $z = +\infty$.

The second section develops the linearized equations for the system (6), (7). The third section describes the normal modes of the system for γ constant. The fourth section considers the special case with no electron bunching and a linear profile for γ. The discussion sums up the results obtained.

2. Formulation of the linearized problem

Before developing the equations for the linearized system associated with (6), (7), we must choose the background state about which to linearize. We select a state with $g \equiv 0$, in which case the general solution of (6) is

$$p(z,\theta) = A(\theta)\exp -\mathrm{i}\{[c + A^2(\theta)]z - \chi(\theta)\}, \tag{9}$$

where $A(\theta)$ is real and positive and $\chi(\theta)$ is real. This solution is consistent with $g = 0$ provided

$$\langle p(z,\theta)\rangle = \frac{1}{2\pi}\oint_0^{2\pi} p(z,\theta)\,\mathrm{d}\theta = 0. \tag{10}$$

We take a particularly simple case with $A(\theta)$ constant, and without loss of generality $A = 1$. For $\chi(\theta)$ we choose

$$\chi(\theta) = \theta + \psi(\theta), \tag{11}$$

where $\psi(\theta)$ is periodic of period 2π. This choice of $\chi(\theta)$ distributes the electron momentum non-uniformly on the unit circle $|p| = 1$. In particular, the arc length of the unit circle between θ_1 and θ_2 is $(\theta_2 - \theta_1) + \psi(\theta_2) - \psi(\theta_1)$, so that the linear density of momentum on the circle is just $\left|1 + \frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right|$. Thus, for $\psi'(\theta) \neq 0$ the electron momentum is bunched on the unit circle. A more general bunching would occur for $A(\theta)$ not constant in θ. We comment on the consequences of this possibility in section 5. If $\psi(\theta) = 0$, the distribution in angle is uniform, and this choice corresponds to the standard initial conditions in numerous calculations, see, e.g. [1, 5]. Finally we linearize the system about the state

$$p_0(z,\theta) = \exp[-\mathrm{i}(c+1)z + \mathrm{i}\theta + \mathrm{i}\psi(\theta)] \tag{12}$$

$$g_0 = 0, \tag{13}$$

where

$$\langle \exp\{\mathrm{i}[\theta + \psi(\theta)]\}\rangle = 0. \tag{14}$$

We discuss possible choices for $\psi(\theta)$ after we determine what properties of $\psi(\theta)$ are relevant to this problem.

We now express a solution of our system as

$$p = p_0[1 + \pi(z,\theta) + \cdots] \tag{15}$$

$$g = [0 + h(z) + \cdots]\exp[-\mathrm{i}(c+1)z], \tag{16}$$

where both π and $h(z)$ are small. To first order in the magnitudes of π and h, the (linearized) system of equations is

$$\frac{\mathrm{d}\pi}{\mathrm{d}z} + \mathrm{i}(\pi + \bar{\pi}) = \mathrm{i}\delta h \exp[-\mathrm{i}\theta - \mathrm{i}\psi(\theta)] \tag{17}$$

$$\frac{\mathrm{d}^2 h}{\mathrm{d}z^2} - 2\mathrm{i}(c+1)\frac{\mathrm{d}h}{\mathrm{d}z} + [\gamma(z) - (c+1)^2]h(z) = \delta\langle \pi \exp[+\mathrm{i}\theta + \mathrm{i}\psi(\theta)]\rangle, \tag{18}$$

where $^-$ above a symbol denotes its complex conjugate.

We can simplify the system substantially with the introduction of

$$u(z) = \langle \pi \exp[\mathrm{i}\theta + \mathrm{i}\psi(\theta)]\rangle, \tag{19}$$

$$v(z) = \langle \pi \exp[-\mathrm{i}\theta - \mathrm{i}\psi(\theta)]\rangle \tag{20}$$

and the constant

$$\rho = \langle \exp[-2\mathrm{i}\theta - 2\mathrm{i}\psi(\theta)]\rangle. \tag{21}$$

The equations for u, v and h are

$$\frac{du}{dz} + i(u + \bar{v}) = i\delta h \tag{22}$$

$$\frac{dv}{dz} + i(\bar{u} + v) = i\delta\rho h \tag{23}$$

$$\frac{d^2h}{dz^2} - 2i(c+1)\frac{du}{dz} + [\gamma(z) - (c+1)^2]h(z) = \delta u. \tag{24}$$

It is straightforward to show that if u, v and h satisfy (22)–(24), then one can reconstruct π to satisfy (17), and the reconstructed π will be consistent with the functions u and v. Thus, the system (22)–(24) is equivalent to the linearized system.

We note that the entire effect of the non-uniformity of the distribution of the electrons on the circle is contained in the constant ρ. By appropriate choice of the phase point $\theta = 0$, we change ρ by multiplication by an arbitrary phase factor $e^{i\theta_0}$. Thus we can assume that ρ is real and non-negative. From the definition it is clear

$$0 \leqslant \rho \leqslant 1. \tag{25}$$

With a few examples of possible choices of $\psi(\theta)$ we can show a number of different physical situations. We know already that if $\psi(\theta) = 0$ then $\rho = 0$. If we set $\theta + \psi(\theta)$ zero in almost the entire range $0 \leqslant \theta \leqslant \pi$ and equal to π in almost the entire range $\pi \leqslant \theta \leqslant 2\pi$, but make $\psi(\theta)$ periodic of 2π, then we can easily satisfy (14), but ρ is close to 1. Thus, while $\rho = 1$ may not be able to be achieved with smooth data, it can be approached arbitrarily closely. This particle distribution corresponds to half the electrons with phase 0 and the other half with phase π, clearly an example of severe bunching.

Two other examples with different forms of bunching would be

$$\psi_1(\theta) = \lambda \cos\theta, \tag{26}$$

or

$$\psi_2(\theta) = \mu \cos 2\theta. \tag{27}$$

In the first case λ is not arbitrary, but must satisfy $J_1(\lambda) = 0$, in order that (14) holds. For $\lambda > 1$ the electron distribution function is not monotone. The largest value of ρ for the distribution $\psi_1(\theta)$, occurs for $\lambda \cong 3.8$ and $\rho \cong 0.18$. For the second case μ may be chosen arbitrarily and the largest value of ρ occurs $\mu \cong 1.5$ and $\rho \cong .48$. Again the electron distribution function is not monotone in θ. These two cases show that the electron distribution functions allow non-monotone 'tongue'-like structures, which are often seen in numerical calculation, see, e.g., [4, 5].

We return to the system (22)–(24), and we find

$$\frac{d}{dz}(u + \bar{v}) = i\delta(h - \rho\bar{h}), \tag{28}$$

so that

$$\frac{d^2u}{dz} = \delta\left(h - \rho\bar{h} + i\frac{dh}{dz}\right) \tag{29}$$

and the wave equation reduces to

$$\frac{d^2}{dz^2}\left\{\frac{d^2h}{dz^2} - i(c+1)\frac{dh}{dz} + [\gamma(z) - (c+1)^2]h(z)\right\} = \delta^2\left(h - \rho\bar{h} + i\frac{dh}{dz}\right). \tag{30}$$

The form of (30) indicates that there is a significant difference between the cases $\rho = 0$ and $\rho \neq 0$. In principle (30) is an eighth-order equation, since it is two coupled fourth-order equations for the real and imaginary parts of h. However in the case $\rho = 0$, if $h(z)$ is a solution then so is $ch(z)$, where c is a complex constant. Thus, if we obtain four solutions of (30) $h_k(z)$ with $\rho = 0$ with the data

$$\left| \frac{\mathrm{d}^j}{\mathrm{d}z^j} h_k(z) \right|_{z=0} = \delta_{jk}, \qquad j = 1, 2, 3, 4', \quad k = 1, 2, 3, 4, \tag{31}$$

then all solutions of (30) are linear combinations of these four. Thus, the polarization of the electromagnetic waves is arbitrary. When $\rho \neq 0$ there are eight distinct solutions and the polarization is not arbitrary and is more complicated. We examine these issues further in the following section where we discuss the more general symmetries that connect the eight solutions.

3. Linear wave propagation

We consider a simple wave propagation problem for the wave equation (30). We treat waves in a uniform medium and identify ‘fast’ and ‘slow’ waves, resonances and cut-offs. We also consider the resonance region in somewhat greater detail. We also comment on the application of geometrical optics methods, and the peculiar nature of the ‘slow’ waves. Much of the analysis is based on the presumed, and plausible, smallness of δ. We start with the case of a uniform medium in which γ is a constant. A discussion of analogous dispersion relations is given in [6] and the solutions were used to describe the gyrotron interaction.

It is clear from the form of (30) with $\rho \neq 0$, that there can be no plane wave solution of the form $h(z) = \exp(\mathrm{i}kz)$. We can, however, look for solutions of the form

$$h = A\exp(\mathrm{i}kz) + \bar{B}\exp(-\mathrm{i}\bar{k}z), \tag{32}$$

where we allow k to be complex. We obtain the algebraic system

$$\{k^2[(k-c-1)^2 - \gamma] - \delta^2(1-k)\}A + \delta^2\rho B = 0 \tag{33}$$

$$\delta^2\rho A + \{k^2[(k+c+1)^2 - \gamma] - \delta^2(1+k)\}B = 0, \tag{34}$$

with the dispersion relation for the wave:

$$[k^2[k^2 + (c+1)^2 - \gamma] - \delta^2\}^2 - k^2[2k^2(c+1) - \delta^2]^2 - \delta^4\rho^2 = 0. \tag{35}$$

The wave dispersion relation is of degree eight. There is, however, a high degree of symmetry in the system. If (A, B, k) represents a solution of (33)–(35) then so does $(B, A, -k)$, $(\bar{A}, \bar{B}, \bar{k})$ and $(\bar{B}, \bar{A}, -\bar{k})$. Of these four solutions the first and fourth are always identical, as are the second and third. When k is real we may assume (A/B) real, in which case all four solutions are identical. However in this special case $\alpha A, \alpha B$, where α is complex, is also a solution. Thus, each real root k corresponds to two waves which differ by multiplication of A and B by a complex constant α. Note that since the wave representation (32) involves A and $\bar{B}$, multiplying A and B by a complex constant is not the same as multiplying the wave by amplitude by a complex constant. Each real wave number k thus corresponds to two waves. We might consider each solution a different ‘polarization’. However, what changes is the phase of the mixture of incoming and outgoing waves. This is not the usual meaning of polarization. Nonetheless, for simplicity we use this term to identify the nature of the degeneracy splitting. When k is a complex number with non-vanishing real and imaginary parts $\bar{k}$ is also a possible wave number, so that if one determines the ratio (A/B) from (33), then $\alpha A, \alpha B$ is also a solution so that four distinct waves are determined. The case k pure imaginary is different in that the form of solution (32) is degenerate, and one must set

$$h(z) = A\exp(\kappa z), \tag{36}$$

and the wave equation becomes

$$[\kappa^2[(\kappa - \mathrm{i}(c+1))^2 + \gamma] - \delta^2(1 - \mathrm{i}\kappa)]A - \delta^2\rho\bar{A} = 0. \tag{37}$$

The dispersion relation is exactly the same as (35) if one sets $k = \mathrm{i}\kappa$, as one would expect. However one now determines the ratio $\bar{A}/A$ from (37), and for a given value of κ two values of $\arg(A)$ are possible. Again a value of k corresponds to two different waves with different phase. Thus the case $\rho \neq 0$ is similar to that with $\rho = 0$ in that each wave number allows two solutions with different, but specific, 'polarizations'. The characterization of the degeneracy as a polarization is at best qualitative.

We can obtain deeper physical insight into the nature of the wave solutions if we examine the solutions (33), (34) using the generally valid assumption $|\delta| \ll 1$. With δ small, but k not small

$$k = c + 1 \pm \sqrt{\gamma} + O(\delta^2) \tag{38}$$

$$B/A = -\delta^2\rho/[4k^3(c+1)], \tag{39}$$

or

$$k = -(c+1) \pm \sqrt{\gamma} + O(\delta^2) \tag{38$'$}$$

$$A/B = +\delta^2\rho/[4k^3(c+1)]. \tag{39$'$}$$

The solutions (38), (39) give rise to exactly the same wave form (32) as the solutions (38′), (39′). We consider only the waves (38), (39) and recall that multiplication of A and B by the complex constant α generates a solution with different 'polarization'. For the solution to be valid $c + 1 \neq \sqrt{\gamma}$ a condition we interpret shortly. The waves are propagating for $\gamma > 0$ and exponentially damped or growing for $\gamma < 0$. Hence $\gamma = 0$ is a wave cut-off. We recall that the physically relevant wave $g(z)$ is related to the wave under study, $h(z)$, by (16) so that these waves correspond to

$$g(z) = A\exp(\pm\mathrm{i}\sqrt{\gamma}z) + \cdots. \tag{40}$$

These are clearly the cavity modes, propagating for $\gamma > 0$, and otherwise damped or growing. We identify these as 'fast' waves, in that $|k|$ is large.

The next group of waves assumes that $k \sim \delta$, and we find easily from (35)

$$k^2 = \delta^2(1 \pm \rho)/[(c+1)^2 - \gamma] \tag{41}$$

$$A/B = \mp 1. \tag{42}$$

We see that the waves propagate for $(c+1)^2 > \gamma$ and otherwise are exponentially growing or damped. Clearly $(c+1)^2 = \gamma$ is a wave resonance for this 'slow' wave. When $c + 1 = \sqrt{\gamma}$ one fast wave and the two slow waves merge. Just as for the fast waves, when 'polarization' is counted there are four slow waves.

It is useful to examine the resonance region near $c+1 \sim \sqrt{\gamma}$. We assume that $(c+1)^2 - \gamma$ and k are small, specifically

$$(c+1)^2 - \gamma = \delta^{2/3}\Gamma \tag{43}$$

$$k = \delta^{2/3}k' \tag{44}$$

and thus

$$-\{(k')^2[\Gamma + 2k'(c+1)] + 1\}A + \rho B = 0 \tag{45}$$

$$\rho A - \{((k')^2[\Gamma - 2k'(c+1)] + 1\}B = 0 \tag{46}$$

so that

$$4k'^6(c+1)^2 - (k'^2\Gamma + 1)^2 + \rho^2 = 0. \tag{47}$$

This dispersion relation valid near resonance couples the two slow waves and the resonant fast wave with small wave number. For these waves A/B is of order 1 in the small parameter δ. It is clear that bunching has a significant effect on the solutions since for all these waves A/B is of order one. Thus, the bunching couples the waves with $k > 0$ and those with $k < 0$. It follows easily from (47) that for $\Gamma > 0$, i.e. $\gamma > (c+1)^2$, there is only one real, positive value of $(k')^2$. This wave for $\Gamma \to 0$ approaches a slow wave, while for Γ large it approaches the fast wave. For Γ large and negative there are three real, positive values of $(k')^2$. One of these is $(k')^2 = -\Gamma/[4(c+1)^2]$ and is the limit of the fast wave, while the other two are limiting forms of the slow wave.

For the following section it is useful to consider the resonance in the case $\rho = 0$, for a system without bunching. We then return to (45) and set $\rho = 0$. The dispersion relation is then

$$2(k')^3(c+1) + \Gamma(k')^2 + 1 = 0. \tag{48}$$

For $\Gamma > 0$ there are no roots with $k' > 0$ and exactly one with $k' < 0$. For $\Gamma < 0$, there is again one root with $k' < 0$, and either two roots with $k' > 0$ or none, according as $1 < (-\Gamma)^3/[27(c+1)^2]$ or $1 > (-\Gamma)^3/[27(c+1)^2]$. Thus, as Γ changes from negative to positive value, there are initially one fast wave and one slow wave with $k' > 0$ and one slow wave with $k' < 0$. The two waves with $k' > 0$ merge and move into the complex plane as Γ increases and the slow wave with $k' < 0$ becomes a fast wave. This description of the wave coupling is exactly the standard one as presented, e.g. in the text of Nusinovich, [2], see p 119 *et seq*. That treatment does not consider the other fast wave, for which $k > 0$, and which is unconnected to the resonance.

After one describes the waves in a uniform medium one could go on to treat a system where γ is not constant, but varies slowly in z, for instance $\gamma = \gamma(\delta z)$. One could generalize (32) and look for solutions of the wave equation of the form

$$h = A(\delta z)\exp[\mathrm{i}\Phi(\delta z)/\delta] + \bar{B}(\delta z)\exp[-\mathrm{i}\bar{\Phi}(\delta z)/\delta]. \tag{49}$$

One would find that $\Phi'(\delta z)$ and A, B satisfy the system (34)–(36). We could apply this analysis except near cut-off and resonance. The representation would be valid for the fast waves, but would not be possible for the slow waves. For slow waves the natural wave number $k \sim \delta$ is also the wave number for which the background state varies. In such a case geometrical optics fails. It is also not clear that the slow waves will be physically relevant. We can see these effects in the explicit solutions of the following section. Only if we were to take $\gamma = \gamma(\delta^2 z)$ and replace δz everywhere in (49) by $\delta^2 z$ could we expect to find a relevant geometrical optics expansion for slow waves. In any case these expansions would fail near cut-off and resonance. In order to complete the geometrical optics expansions we would need connection formulae across resonance and cut-off. However, the validity of the method of matching asymptotic expansions near cut-off and resonant is also compromised by the complicated nature of the slow waves. In the following section we obtain uniformly valid solutions, but only for the case $\rho = 0$, without particle bunching. We can also identify the minor role the slow waves play in the gyrotron.

4. An exact solution without particle bunching

In this section we obtain integral representation of the solutions of the wave equation (30) for the case of no particle bunching, $\rho = 0$, and for a simple form for $\gamma(z)$. The form we choose,

$$\gamma(z) = \beta\delta z \tag{50}$$

allows us to obtain connection formulae for solutions across cut-off and resonance. With some effort we might treat a more general form for γ, $\gamma = \gamma(\delta z)$, and by the use of matched asymptotic expansions we could attempt to construct a solution. We content ourselves in this paper with the construction of the solutions of the wave equation (30) with γ given by (50).

We look for solutions by the method of integral transforms, and we seek a solution

$$h(z) = \int_C \mathrm{d}s \exp(sz) H(s), \tag{51}$$

where the contour and the analytic function $H(s)$ are unknown. We assume that the integrand in (51) tends to zero rapidly near the end points of the contour C. If $H(s)$ satisfies the ordinary differential equation

$$s^2 \left\{ [s - \mathrm{i}(c+1)]^2 - \beta\delta \frac{\mathrm{d}}{\mathrm{d}s} \right\} H(s) = \delta^2 (1 + \mathrm{i}s) H(s), \tag{52}$$

and if the conditions near the end points of C are met, then $h(z)$ satisfies the wave equation. We find easily

$$H(s) = \exp\left[\left\{\tfrac{1}{3}[s + \mathrm{i}(c+1)]^2 + \delta^2\left(\tfrac{1}{3} - \mathrm{i}\log s\right)\right\}/(\beta\delta)\right], \tag{53}$$

and we need only select the contours of integration. We note that for $|s|$ large

$$H(z) \sim \exp[s^3/(3\beta\delta)] \tag{54}$$

while for $|s|$ small

$$H(z) \sim \exp[\delta/(s\beta\delta)]. \tag{55}$$

Thus, contours of integration may end at infinity provided Re $(s^3) < 0$ there or at the origin provide Re $s < 0$.

We define four solutions, $h_i(z)$, $i = 1, 2, 3, 4$ by giving four contours, C_i, $i = 1, 2, 3, 4$, in figure 1. It is easy to conclude that the integrand of (51) vanish exponentially fast on all the contours of integration in the neighborhoods of infinity and the origin. Thus, $h_i(z)$, $i = 1, 2, 3, 4$ are solutions of the equation. Once we have shown that the functions are solutions it is convenient to move the contours of integration for h_2 and h_3 to the negative and positive imaginary axes, respectively. For $h_4(z)$ we may use the contour of figure 2. We should show that the four solutions are linearly independent. We sketch only a few points in such a proof. First, $h_1(z)$ is exponentially large in γ/δ for $\gamma < 0$. The other functions are bounded in $\gamma < 0$, hence $h_1(z)$ is linearly independent of the other three. The asymptotic expansions of h_2, h_3 and h_4 in δ, which we give shortly, show that these three are linearly independent. Hence we have constructed four linearly independent solutions.

To explore the nature of the solutions just found, we may rewrite (51) as

$$h_i(z) = \int_{C_i} \mathrm{d}s \exp[E(s, \delta)/(\beta\delta)], \tag{56}$$

where

$$E(s, \delta) = s\gamma(z) + \frac{1}{3}[s - \mathrm{i}(c+1)]^3 + \delta^2\left(\frac{1}{s} - \mathrm{i}\log s\right). \tag{57}$$

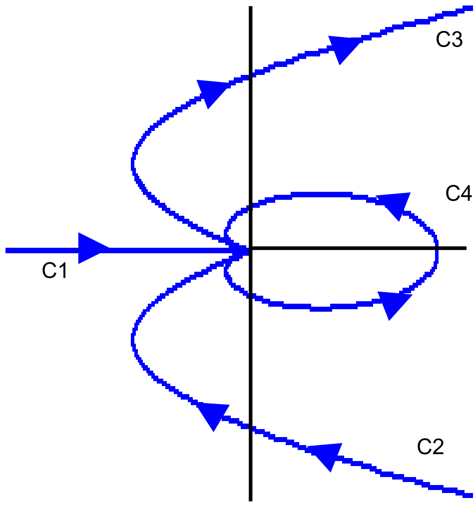

Figure 1. Contours of Integration.

In order to obtain asymptotic expansions of $h_i(z)$ for δ small we must examine the stationary points of $E(s, \delta)$ in the variable s. These stationary points satisfy

$$\gamma + [s - \mathrm{i}(c+1)]^2 - \delta^2 \left(\frac{1}{s^2} + \frac{\mathrm{i}}{s}\right) = 0, \tag{58}$$

and if we set $s = \mathrm{i}k$ we see that k satisfies the dispersion relation obtained from (33) when ρ is set to zero. Thus, we know that there are fast wave roots with $k \sim 1$ given by (36), and these roots represent exponentially damped or growing solutions for $\gamma < 0$ and purely oscillating solutions for $\gamma > 0$ and away from resonance. There are also two slow wave roots with $k \sim \delta$, and away from resonance these roots represent oscillating solutions for $\gamma < (c+1)^2$ and solutions with k imaginary for $\gamma > (c+1)^2$. We have already examined the resonance region with the scaling (43) and we found that there is always one root corresponding to a wave with $k < 0$ and depending on Γ there are two real roots or no real roots with $k > 0$. In all, there is always exactly one root with $k < 0$, and one, two or three roots with $k > 0$.

It is important to note that with the contours of integration for h_2 and h_3 on the imaginary axis, $E(s, \delta)$ is pure imaginary there plus a real constant $\pm(\pi/2)\delta^2$. Thus, one can obtain asymptotic expansions for small δ of the solutions h_2 and h_3 purely by the method of stationary phase. In particular, any roots of (58) off the imaginary axis are irrelevant. Thus, for these two solutions the asymptotic expansion consists of propagating waves plus some presumably small error terms. For these two solutions when $\gamma < 0$ there is at most a slow wave contribution plus small error terms. We indicate shortly that the slow wave contribution is no larger than the error, so that h_2 and h_3 represent incoherent wave motion for $\gamma < 0$. Between resonance and cut-off h_3 consists of the two fast waves, incoming and outgoing, given by (36), while h_2 is again an incoherent wave. We discuss the resonance region shortly. Above resonance, $\gamma > (c+1)^2$, h_2 consists of an outgoing fast wave plus corrections, while h_3 is an incoming fast wave plus corrections.

We next turn to examine h_4, and with the change of variable from s to t where

$$s = t\delta \tag{59}$$

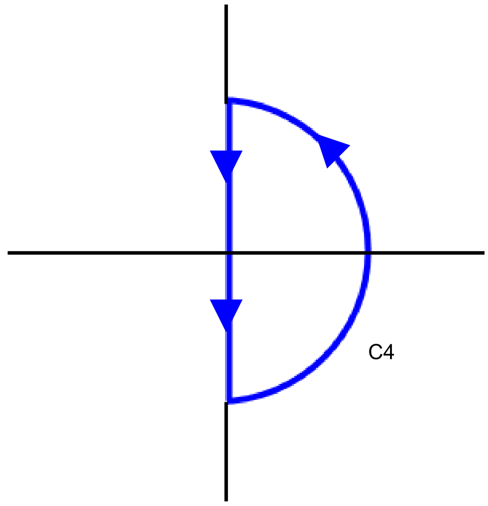

Figure 2. A modified contour of integration.

we find

$$h_4(z) = \delta \exp\left[\frac{\mathrm{i}(c+1)^3}{3\beta\delta} - \frac{\mathrm{i}\log\delta}{\beta}\right]\int_{C_4} \mathrm{d}t \exp\left[\frac{\tilde{E}(t,\delta)}{\beta}\right] \tag{60}$$

where

$$\tilde{E}(t,\delta) = \left\{t[\gamma - (c+1)^2] + \frac{1}{t} - \mathrm{i}\delta\log t + \frac{\delta^2 t^3}{3} - \mathrm{i}\delta t^2(c+1)\right\}. \tag{61}$$

We see from (60), (61) that $h_4(z)$ can be expanded as a power series in γ, which converges for all γ, and if we pick the semi-circle of C_4 to have a radius of $\sqrt{\gamma}$, then

$$|h_4(z)| \leqslant \delta k \exp(L|\sqrt{\gamma}|), \tag{62}$$

where K and L are independent of δ and of order one. We also see from (60) that $h_4(z)$ is no smaller than $O(\delta)$. Clearly h_4 represents an incoherent wave of approximately the same amplitude in the entire region of interest. A similar change of variable of integration (59) for h_2 and h_3 would show that there is no slow wave in h_2 or h_3 and that the contribution to h_2 and h_3 from the region $s \sim \delta$ is itself $O(\delta)$. Thus, the incoherent wave structures in h_2 and h_3 are $O(\delta)$. This analysis indicates that away from resonance the slow waves cannot be easily identified and are lost in incoherent waves.

The solution h_1 is exponentially large below cut-off, where $\gamma < 0$, and decreases in magnitude as γ increases. We find easily that

$$h_1 \sim \kappa_1\sqrt{\delta}|\gamma|^{-1/4}\exp\left\{\left[\mathrm{i}(c+1)\gamma - \tfrac{2}{3}\gamma\sqrt{|\gamma|}\right]/\delta\right\}, \tag{63}$$

where κ_1 is a constant of order one in magnitude. For $\gamma > 0$ one finds that an estimate of the form (62) applies, but with $L < 0$. Thus, the solution h_1 is exponentially large in δ for $\gamma < 0$ and decreases exponentially as γ increases up to cut-off, while for $\gamma > 0\, h_1$ is $O(\delta)$.

We can now construct a representation of a wave in the gyrotron, which is an outgoing wave beyond resonance

$$h(z) = h_2(z) + \eta_1 h_1(z) + \eta_4 h_4(z), \tag{64}$$

where η_1 is chosen so that $|\eta_1 h_1(z)| \sim \delta$ for the most negative value of γ in the cavity and η_4 is of order one. Well below resonance this solution is an incoherent wave of amplitude $O(\delta)$, while above resonance

$$h(z) \sim \sqrt{\delta} K' \gamma^{-1/4} \exp\left\{\left[\mathrm{i}(c+1)z - \tfrac{2}{3}\mathrm{i}\gamma^{3/2}\right]\big/(\beta\delta)\right\}, \tag{65}$$

where K' is a constant of order one in magnitude. Thus, acceptable solutions are not unique, as η_1 and η_4 are essentially arbitrary. Further, the amplification of the wave amplitude across resonance is $\delta^{-1/2}$.

As a consequence of these observations we conclude that no coherent wave motion is observable until one moves into the resonance region where (43) must apply, or $(c+1)^2 - \gamma = O(\delta^{2/3})$. In this region a slow wave emerges from the noise and converts into a fast wave as one moves across resonance. One can easily give the asymptotic expansion analogous to (65) which is valid in this transition region and one finds

$$h(z) \sim \frac{\sqrt{\delta} K'' \exp[\mathrm{i}(\kappa\Gamma - (1/\kappa)(\delta^{1/3}/\beta)]}{[|\kappa|^{-3} + (c+1)]^{1/2}}, \tag{66}$$

where κ and Γ are given by (48) and (43), respectively and $\kappa < 0$. The wave represented by (65) is exactly the mode which is slow for Γ large and negative and fast for Γ large and positive.

If one considers the effects of noise in a numerical solution of the linearized wave equation, no solutions grow exponentially in $(1/\delta)$ and other than the emergence of the outgoing fast wave in h_2, all the other solutions remain of the same order of magnitude as their initial data provided $|\gamma|$ is bounded. Thus, within the linear regime substantial error magnification does not occur.

Two remaining questions are: Why is there no wave which grows exponentially as γ increases from a negative value of large magnitude, corresponding to the second root of (36)? And why is no slow wave observable? We start with the simpler, second question. We have already noted that with the scaling chosen the background state varies in z on the same distance scale as the slow wave. Thus, it is not surprising that the slow wave is lost in the noise. We could change $\gamma(z)$ from $\gamma(\delta z)$ to $\gamma(\delta^2 z)$ easily if we set $\beta = \delta$. In this case if we examine h_4, and (60), (61), we see that there is indeed a contribution from the stationary points of $\tilde{E}(t, \delta)$ from the slow waves. The amplitude of these waves would be $O(\delta^{3/2})$, while the amplitude of the fast waves would be $O(\delta)$. Thus, the slow waves may be present, but with relatively small amplitude. We conclude that the relevance and appearance of the slow waves depend sensitively on the actual parameters in the problem.

To look for exponentially growing waves as γ increases from a large negative value we consider the solution

$$h_5 = h_2 + h_3 + h_4. \tag{67}$$

We note that above cut-off, but away from resonance, this solution is a mixture of incoming and outgoing waves of essentially the same amplitude. We may express h_5 as an integral along a contour in the s plane along a line $s = \tau + \mathrm{i}\sigma$, $-\infty < \sigma < \infty$, where τ is real and positive. For $\gamma < 0$ it is then easy to obtain an expansion for δ small of the form (63) with γ replaced by $-\gamma$. This solution clearly grows exponentially as $-\gamma$ decrease to zero. Thus, the exponentially growing mode is present, but is not useful in constructing an outgoing wave as it is a linear combination of incoming and outgoing waves of the same amplitude.

5. Discussion

This paper continues the analysis of a simple model system of equations for a gyrotron. In addition to the limitations inherent in the model, this work treats only a linearized approximation to the system. Nonetheless, linear solutions often give some indications as to the qualitative behavior of solutions of the full, nonlinear problem. Our examination of this linear problem is somewhat more general than what is usually done in that we consider a more general background state. Normally one assumes that the electrons are uniformly distributed in gyrophase angle. Although this is a reasonable assumption for upstream, simulations demonstrate the essential role of electron bunching. We choose a background state with particle bunching allowed. We do assume, however, that the momentum of the background state in the direction perpendicular to the static magnetic field is a constant independent of gyrophase angle. If we were to give up this hypothesis, the entire structure of the problem would change dramatically, and the resulting linearized ordinary differential equations would be profoundly more complicated. In particular, one could not find a simple system of equations equivalent to (22)–(24). While bunching should also affect the perpendicular momentum we are unable to treat this problem easily, and we content ourselves with the limited role of bunching we can easily accommodate. We find that in the present linearized case the entire effect of bunching occurs through one explicit parameter.

We look first at the waves present in the uniform state with bunching. We obtain a more intricate dispersion relation than is usually seen. Propagating waves couple both incoming and outgoing components, although each wave is dominantly either outgoing or incoming. An incoming or outgoing wave exists in the form of two 'polarizations'. As used here polarization refers to the mixture of incoming and outgoing waves. We find, away from resonance, four cavity modes, incoming and outgoing, each with two 'polarizations'. We identify these as fast modes. There are also four slow modes, incoming and outgoing with two 'polarizations'. We give expansions of these modes in terms of a natural small parameter of the system. Near resonance one of the fast waves has a small wave number and is indistinguishable from slow modes. In the simpler case with no bunching one can follow the changes in the modes as one varies the natural wave number of the cavity. There is always exactly one outgoing wave, which for γ less than resonance is a slow mode with low wave number. This wave becomes a fast wave for γ above resonance. The three other waves correspond to one incoming fast wave and two exponentially growing or decreasing slow waves for γ above resonance. Between resonance and cut-off there are two slow waves, one outgoing and just described and one incoming, and two fast waves, one incoming and one outgoing. Below cut-off the fast waves grow or decay exponentially in $1/\delta$ while the slow waves are largely unaffected.

In a slowly varying non-uniform state one could construct a geometrical optics-like theory. Such a theory would apply away from cut-off and resonance. The theory applies naturally for fast waves. The application for slow waves is far more problematic as the natural wave number of the waves could easily match the natural wave number associated with the spatially varying background state. We see this effect far more explicitly when we examine the full solution of a wave propagation problem for a case with no particle bunching and a simple profile for the variation of the cavity wave numbers. The construction of solutions for this problem might also allow the use of the method of matched asymptotic expansions to construct solutions for more general variations of the cavity wave number. The validity of the process is very likely highly sensitive to the actual parameters of the problem of interest.

With no particle bunching and a simple profile function of the square of the wave number, which is a linear function of the distance, one can construct four linearly independent solutions of the underlying fourth-order ordinary differential equation. One can then examine the nature of the solutions in terms of expansions in the natural small parameter by the method of stationary phase. Of the four solutions one involves an integral over purely positive wave numbers and another is an integral over purely negative wave numbers. Another is a function that is a convergent power series in $\gamma(z)$ and which is bounded by the form $K \exp(L\sqrt{\gamma})$

for constants K and L. The fourth solution becomes exponentially large below cut-off in the small parameter δ and grows as $K' \exp(\sqrt{\gamma}/\delta)$. Clearly this fourth solution cannot be used to construct a relevant solution to the gyrotron equation. To obtain an outgoing wave we would choose a linear combination of the solution with only outgoing waves plus two other solutions. We see immediately that there is no unique acceptable solution to this problem. Further, for the linearized system the outgoing wave solutions are stable in the sense that if one adds at one point small admixtures of other solutions, the modified outgoing wave solution will only change everywhere downstream by small amounts. A more detailed analysis also indicates that below resonance there is no clear, well-defined wave pattern. Only near resonance does the fast wave emerge from the 'noise'. Finally the amplitude of the fast wave increases by $O(1/\sqrt{\delta})$ as the wave crosses resonance.

The analysis also indicates the difficulties inherent in numerical solution of this problem in the use of a 'shooting' method in which initial data is given for some value of z corresponding to a negative value of γ. There is a solution which grows exponentially starting at that value of γ; it is given by (67). However, it does not correspond to a purely outgoing wave. Such initial data was incorrectly proposed in [1]. As has been indicated, well below resonance there is no clear wave structure to the desired gyrotron solution, and there are many acceptable solutions. Thus, the correct choice of initial data is a largely indeterminate problem.

Although this paper is primarily an examination of the properties of the solutions of a model gyrotron equation the results have some consequences which could be compared with experiment. The explicit predictions given here relate to the appearance or non-appearance of the slow wave before resonance and an exponentially growing mode amplitude below cut-off on the one hand, and the possible coupling of incoming and outgoing waves above resonance as a result of particle bunching. To compare the results and experiment it is necessary to recognize that the original system (6), (7) has two parameters c and δ, and one free function $\gamma(z)$, which we took of the form $\gamma(\beta\delta z)$. The parameter c is of order one and does not enter substantially in our discussion provided $c > -1$. We assume δ is small, but the magnitude of β enters significantly into the discussion. That is, the magnitude of the beam-wave coupling δ, and its comparison with the scale length of the cavity free space wave number $\beta\delta$ strongly affect the properties of the solution below resonance. This qualitative claim is already susceptible of experimental check.

The determination of the roots of the dispersion relation with some particle bunching is straightforward. The question is which of the solutions obtained are observable? The explicit solutions of the linearized equations in section 4 indicate that in no case are exponentially growing mode amplitudes present below resonance, see the discussion following (64) to the end of the section, and unless β is small—a relation between the cavity wave number and strength of the interaction—no slow wave is observable until very close to resonance. One should be able to observe such effects by varying the cavity design. When β is of order one it should be difficult to observe any consistent wave structure below resonance. When β is small, a slow wave of small amplitude should be visible. Above resonance the slow waves may grow exponentially, but over large distances of order $1/\delta$ the amplification is small, so the growth should not destroy the outgoing wave. However, the analysis of the dispersion relation with bunching and the explicit solutions of section 4 suggest that it should be difficult to produce a pure outgoing wave. Bunching will tend to mix in some small amplitude incoming wave and there will always be some residual noise in the solution. It is possible that the noise, indicated by the form of the possible solution (64), will be large enough to hide the incoming wave. Again, experimental variation of the parameter β should be able to show the changes in these effects.

It remains to be seen how much of this analysis can be extended to the nonlinear problem. This matter will be addressed in a subsequent publication.

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